Lecture 5 - Andreas Rosén 15/06/2015.

(1) Bilinear Hilbert transform:

$$(f_1,f_2) \longmapsto \text{p.v.} \int f_1(x+\beta_1 t)\, f_2(x+\beta_2 t)\, \frac{dt}{t}.$$

(2) Dual trilinear form.

$$(f_1,f_2,f_3) \longmapsto \text{p.v.} \int_{\mathbb{R}}\int_{\mathbb{R}} \prod_{i=1}^{3} f_i(x+\beta_i t)\, \frac{dt}{t}\, dx.$$

$L^{p_1}\times L^{p_2}\times L^{p_3}$. $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. $2<p_i<\infty$.

Can consider (1) to be higher dimensional → i.e. $x,\beta_i\in\mathbb{R}^2$.

~~Example~~ (3): triangular Hilbert transform.

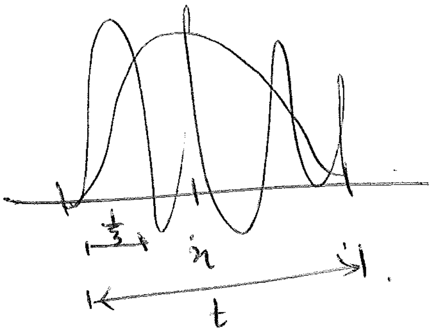
$$(f_1,f_2,f_3) \longmapsto \iiint \frac{f_1(x,y)\, f_2(y,z)\, f_3(z,x)}{x+y+z}\, dx\, dy\, dz.$$

How did we prove (2):

· $|\Lambda(f_1,f_2,f_3)| \lesssim \prod_{i=1}^{3} \|f_i\|_{L^{p_i}}$.

time frequency transform: $f(x)\longmapsto F(x,\xi,t)$

$$= \int f(y)\,\frac{1}{t}\,\varphi\!\left(\frac{y-x}{t}\right) e^{i\xi(y-x)}\, dy.$$



Need: $\|F\|_{L^{2,\infty}(\mathbb{R}^3_{+,x,\xi})} \lesssim \|f\|_2$

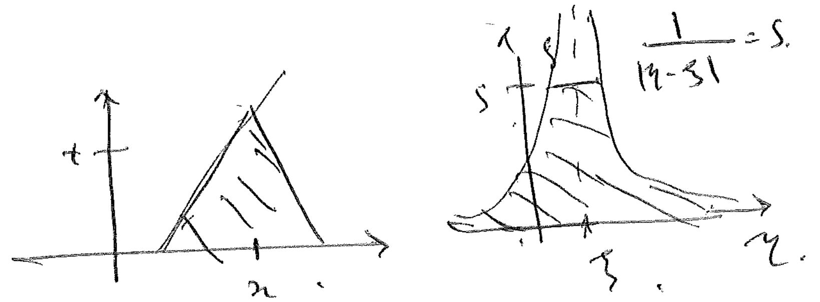
After estimate:

$$\left|\iiint \prod_i F_i(x, \alpha; \xi+\beta_i, t, t)\, dx\, d\xi\, d\eta\right| \lesssim \Big\|\prod_i F_i\Big\|_{L^1(\mathbb{R}^3_+, \sigma, S)} \lesssim$$ (outer Hölder)

$p=\infty$: last time.

$p=2$: <u>today.</u>

$\mathbb{E} = \{\text{true freq. tents.}\}$.

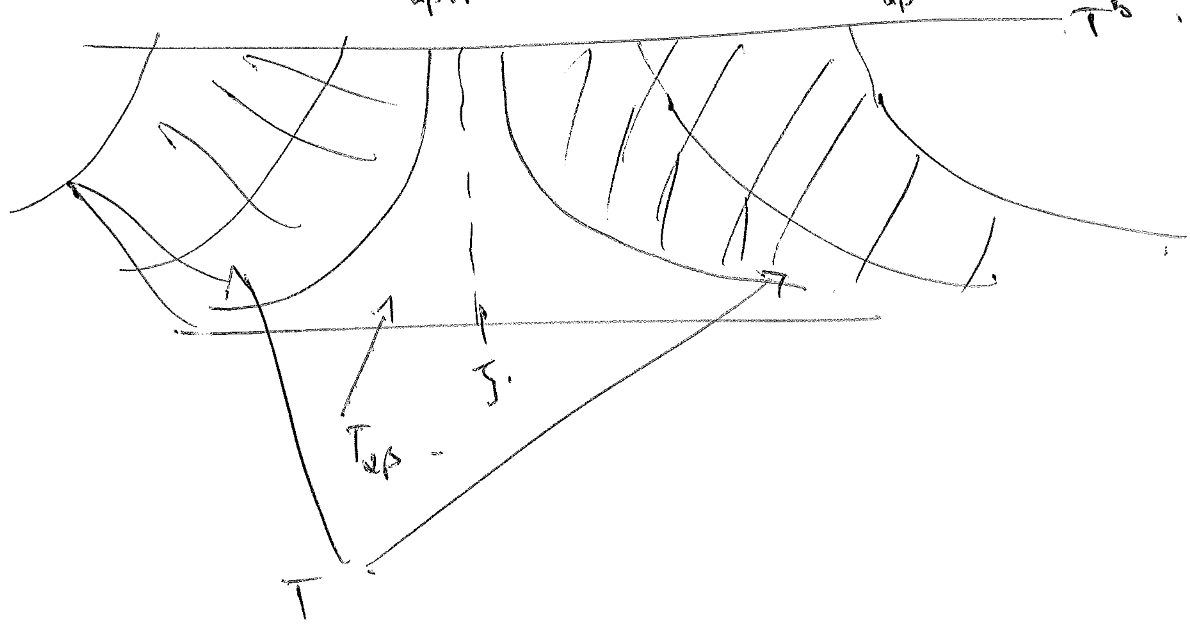


discrete collection of tents $\mathbb{E}_\Delta \subset \mathbb{E}$ requiring:

$$(x, \xi, t) = (2^{k-4} n,\ 2^{-k-8} \ell,\ 2^k), \quad n, \ell, k \in \mathbb{Z}.$$

$\sigma(T(x,\xi,t)) := t \rightsquigarrow \mu$ outer meas.

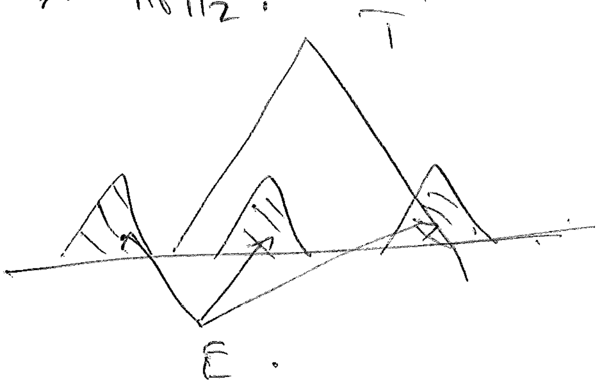
$$S^b(F,T) = \Big(\frac{1}{t}\int_{T_{\alpha\beta}\setminus T^b} |F|^2\, dy\, d\eta\, ds\Big)^{\frac{1}{2}} + \sup_{T_{\alpha\beta}} |F|.$$



Need: Given $\lambda > 0$, construct a family of tents $T \in Q \subset \mathbb{E}_\Delta$ s.t. $E = \bigcup_{T \in Q} T$

$$\Rightarrow \quad S^b(F 1_{X \setminus E}, T') \le \lambda, \quad \forall T' \in \mathbb{E}_\Delta \quad \text{and}$$

$$\sum_{(x,\xi,t) \in Q} t \lesssim \lambda^{-2} \|f\|_2 .$$



$$\mu(S(F) > \lambda) \lesssim \frac{1}{\lambda^2} \|f\|_2 .$$

Pf

Assume $\|f\|_2 = 1$, and spt $\hat{f}$ is compact.

(1) Tents when "$S^\infty(F)$ is big."

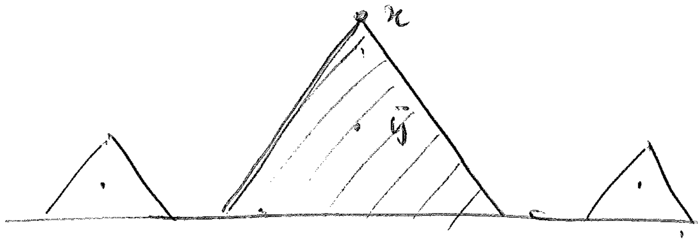
$Q = Q_0 \cup Q_+ \cup Q_-$ $\leftarrow$ split collection this way. (to be done later!)

Assume tents $T_1, \dots, T_{n-1}$ chosen. Choose ~~$(y_n, \eta_n, s_n) \in$~~

$\bar{y} := (y_n, \eta_n, s_n) \mathring{\in} T(x_n, \xi_n, t_n) =: \bar{x}$ $\leftarrow$ dyadic

non-dyadic

s.t.
$$\begin{cases} (y_n, \eta_n, s_n) \in \mathbb{R}^3_+ \setminus (T_1 \cup \dots \cup T_{n-1}). \\ |F(y_n, \eta_n, s_n)| > \lambda. \\ t_n \text{ is maximal.} \end{cases}$$



$\rightsquigarrow$ Define the collection Q_0 of tents.

(1b). Estimates for Q_0: Need $\sum_k t_k \lesssim \lambda^{-2}$.

- Boils down to almost orthogonality argument.
- Schur estimate technique.

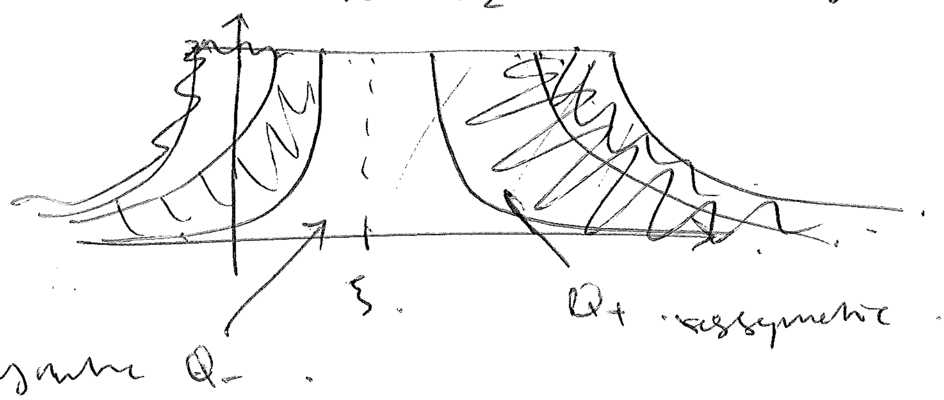
T_k at $x_k \rightsquigarrow \varphi_k(y) = \frac{1}{t_k}\varphi\left(\frac{y-x_k}{t_k}\right)e^{i\xi_k(y-x_k)}$.

Need $|\langle \varphi_k, \varphi_\ell\rangle| \lesssim$ good.

- Combinatorial argument: if $\langle \varphi_k, \varphi_\ell\rangle \neq 0$, then

$[x_k - 2^{-8}t_k, x_k+2^{-8}t_k] \cap [x_\ell - 2^{-8}t_\ell, x_\ell + 2^{-8}t_\ell] = \emptyset$.

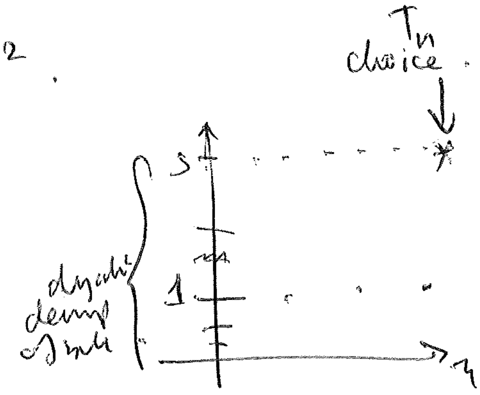
(2). Tents where "$S_2(F)$ is big".



Selection of Q_+: Assume $T_1, \dots, T_{n-1}$ to be chosen.

Consider $(x,\xi,t) \in X_\Delta$ s.t.

$$\frac{1}{t}\iint_{T_{\Delta\beta}(x,\xi,t)\cap X^+_{\xi}\cap (T^b(x,\xi,t))^c\cap E^c_{n-1}} |F|^2\, dy\, d\eta\, ds \geq 2^{-8}\lambda^2.$$



$E_{n-1} = E_0 \cup T_1 \cup \dots \cup T_{n-1}$.
from step 1.

Choose T_n amont these s.t. ξ_n is maximal and t_n maximal.
$T(x_n, \xi_n, t_n)$.

(2b) Estimates:

$$\sum t_{k} \lesssim \lambda^{-2}.$$

- Almost orthogonality argument. (Schur estimate).

"$\sup_{k} \sum_{l \geq k} |\langle \varphi_k, \varphi_l \rangle|$."

- combinatorial argument (similar in spirit to before).

(3). Mutatis mutandis for left parts, and set $Q := Q_0 \cup Q_+ \cup Q_-$, and

$$\Gamma := \underbrace{\bigcup_{T \in Q_0} T}_{L^\infty \text{ big}} \cup \underbrace{\bigcup_{T \in Q_+} T}_{\text{right } L^2 \text{ big}} \cup \underbrace{\bigcup_{T \in Q_-} T}_{\text{left } L^2 \text{ big}}.$$

Note: (1) – (3) is "like" a stopping time argument. But unlike classical stopping time argument, we don't descend recursively into boxes (or trees).

But use a Calderón-Zygmund cleanup to isolate bad parts.